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A SYNTHESIS OF CRACK GROWTH THEORIES IN FATIGUE INTO SOME STATISTICAL DISTRIBUTIONS OF FATIGUE LIFE

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A SYNTHESIS OF CRACK GROWTH THEORIES IN FATIGUE INTO SOME STATISTICAL DISTRIBUTIONS OF FATIGUE LIFE

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The proposed model for fatigue life is a synthesis of the distributional properties of three separate phases of fatigue crack growth. The convolved distribution of life contains three estimable parameters and a function disposable within a restricted class. The problem of estimating all three parameters using the method of maximum likelihood is examined and shown to be impossible of solution in certain cases. This indicates that one of the parameters must be determined *a priori* from concepts of strength of materials or fracture mechanics.

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PREFACE

The research reported herein was conducted under Air Force Contract F 33615-73-C-4016 at Washington State University. The work was initiated under project 7071. The technical monitor of the contract was Dr. H. Leon Harter, Aerospace Research Laboratories, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio. The principal investigator was Professor Sam C. Saunders. The author acknowledges helpful conversations on the problems of fatigue crack initiation and growth and on the practical utility of mathematical techniques with Messrs. Dan Whittaker, Rao Varanasi and Joseph Butler of the Fatigue Research Group of the Boeing Commercial Airplane Company.

This interim report covers the work conducted during the period from September 15, 1973 until September 15, 1974. The manuscript was submitted for publication in March 1974.

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SECTION I

INTRODUCTION

One of the ubiquitous problems in the assessment of fatigue life is the determination of the "strength" remaining when structural components have been subjected to a specified stress regime for a known length of time.

Even in the simplified circumstances when nominally identical structural components are subjected to only an alternating stress of a given maximum value with fixed stress variation (such as might be the case for a helicopter rotor) the fatigue life will still exhibit sufficient variability that the mathematical model deemed most suitable is that the fatigue life is a random variable with an appropriate distribution. Of course, the resolution of the question of which parametric class of distributions is the most useful and appropriate has not been universally agreed upon.

A concept which has historical precedent is that the characteristic life β of a structural component, which has been selected by chance and then subjected to repeated applications of a maximum stress level s, can be expressed as a function of that stress, say

$$\beta = K(s) \tag{1}$$

where K is a decreasing function, usually assumed to be known except for some parameters.

The equation (Eq (1)) is called the Wöhler equation and it expresses the characteristic life in cycles to failure as a function of the maximum stress. In statistical parlance this is a regression equation. The plot of sample data used to estimate this relation is called the statistical S-N diagram i.e. stress versus number of cycles to failure. Various specifications have been adopted for K. One, for example, is the Lündberg formula

$$K(s) = \left(\frac{b}{s-s_0}\right)^{\frac{1}{a}} \qquad \text{for } s > s_0 \tag{2}$$

where s_0 is the value of the stress called the "fatigue limit" and the constants $a, b > 0$ are determined from the material. These parameters are usually estimated by least squares, or by eye, from the statistical S-N diagram.

One approach is to specify a life length, before which failure will occur with preassigned probability P according to the postulated distribution of life, and to construct a P-S-N diagram (see Bastenaire [1] for notation and terminology) in accord with a cumulative damage hypothesis, such as Miner's Rule, using equi-probability curves.

Miner's Rule utilized with the Wöhler regression equation would be as follows: In a loading spectrum which contains n_i cycles of maximum stress level s_i for $i = 1,\ldots,k$ the characteristic life is given (in number of spectra which can be repeated) by the reciprocal of

$$\sum_{i=1}^{k} \frac{n_i}{K(s_i)} .$$

This brief historical perspective is presented to point out how the current approach attempts to overcome, or strengthen, the weakness of the theory previously employed.

The first point is that the approach just mentioned does not recognize the decided influence upon fatigue life which the load order within the spectrum plays. This load order influence has been known to research workers for a long time, see Ref [2], but the implementation of such knowledge

into practical formulae useable in preliminary airframe design has not been accomplished.

The second point is that there is a great danger in assuming a theory which has as a necessary component a so-called "fatigue limit" i.e. a value below which, if the stress is reduced, the fatigue life becomes infinite. Such a theory does not take into account flaws, accidents or "the thousand natural shocks that metal (and flesh) is heir to" which can and do initiate fatigue failure.

SECTION II

SOME CONSIDERATIONS IN THE DERIVATION OF LIFE LENGTH MODELS FOR FATIGUE

The simple model which was originally utilized to obtain the distributions derived in Ref [3] was as follows: During the stable, microscopic phase of fatigue crack growth, following crack initiation, the incremental crack extension during each repetition of the loading spectrum was postulated to be a random variable. The sum of a given number of such variables represented the total crack length after the repetition of a number of spectra. Failure takes place when for the first lead cycle the total crack length reaches a critical value.

We assume that the incremental crack extensions per cycle are independent and identically distributed variates. If X_i for $i = 1,\ldots,n$ are the increments, each with mean μ and variance σ^2, then after n such cycles, with $S_n = X_1 + \ldots + X_n$ the total crack length, the random number N of such increments before failure, i.e. before reaching a critical crack length w, must satisfy

$$P[N \leq n] = P[S_n \geq \omega] = P\left[\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \geq \frac{\omega - n\mu}{\sqrt{n}\,\sigma}\right] .$$

Hence we have asymptotically, by the central limit theorem,

$$F_N(n) \cong \Phi\left[\frac{n\mu - \omega}{\sqrt{n}\,\sigma}\right]$$

where Φ is the standard normal distribution. In this form, this distribution had been discovered by Freudenthal and Shinozuka in [4], before its

publication in [3]. A few years before that it had been given by E. Parzen [5]. Moreover, T. Von Karman had found it in the twenties. (One is led to examine the collected works of C. F. Gauss in this regard.)

The difference between the approach in [3] and the others was the interpretation and parameterization. Let

$$\beta = \frac{\omega}{\mu}, \quad \alpha = \sigma/\sqrt{\mu\omega} .$$

Then the distribution can be written in the form

$$\Phi\left[\frac{1}{\alpha}\xi(t/\beta)\right] \qquad \text{for} \quad t > 0$$

where β is the median life and α is a shape parameter and where

$$\xi(x) = \sqrt{x} - \frac{1}{\sqrt{x}} \qquad \text{for} \quad x > 0 .$$

Moreover, estimation procedures have been derived in [6] for this distribution. This distribution is related to the log-normal in that it satisfies a functional relationship

$$\xi(x) = -\xi\left(\frac{1}{x}\right)$$

which is a relationship between a variate and its reciprocal. We now ask: Is it possible to obtain, by other simple plausible models which count cycles until failure, some useful alternative distributions which are within or similar to the original class?

Let us first suppose that the critical value of the crack length at which the structure fails is a normal random variable W with mean ω and variance ρ^2.

Let a critical value w be given. Then the number N of cycles to exceed w has, by the preceding argument, the distribution

$$P[N \le n | w] = 1 - \Phi\left(\frac{w-n\mu}{\sigma\sqrt{n}}\right) .$$

Using the law of total probability we find the distribution

$$P[N \le n] = \int_{-\infty}^{\infty} \Phi\left[\frac{n\mu-w}{\sqrt{n}\,\sigma}\right] d_w\, \Phi\left(\frac{w-\omega}{\rho}\right)$$

$$= \Phi\left(\frac{n\mu-\omega}{\sqrt{n\sigma^2+\rho^2}}\right) .$$

Let $\beta = \frac{\omega}{\mu}$, $\frac{1}{\alpha} = \sqrt{\frac{\mu\omega}{\sigma}}$, $\varepsilon\beta = \frac{\rho^2}{\sigma^2}$

and we have

$$F_N(n) = \Phi\left[\frac{1}{\alpha}\,\xi_1\left(\frac{n}{\beta}\right)\right]$$

where

$$\xi_1(t) = \frac{\xi(t)}{\sqrt{1+\frac{\varepsilon}{t}}} \qquad \text{for} \quad t > 0$$

where $\varepsilon > 0$, small, is a nuisance parameter. However, note that

$$\lim_{t \to s_\infty} \xi_1(t) = \infty , \qquad \lim_{t \to 0} \xi_1(t) = -\frac{1}{\sqrt{\varepsilon}} .$$

This value determines the probability that the structure is broken (failed) at time zero. (Such an event has been known to occur in practice). As a consequence, ξ_1 as given cannot satisfy the reciprocal relationship, to wit, $\xi_1(t) = -\xi_1\left(\frac{1}{t}\right)$.

Let us also examine further the second assumption concerning crack growth made in [3]; namely, given the total crack length at the start of the loading cycle is s, the incremental extension is a normal variate with mean $\mu + \delta s$ and variance σ^2.

From the results given in [3] we know that the random number of cycles to exceed the critical length ω is

$$F_N(n) = \Phi\left[\frac{\mu_n - \omega}{\sigma_n}\right]$$

where

$$\mu_n = \frac{\mu}{\delta}\left[(1+\delta)^n - 1\right], \quad \sigma_n^2 = \sigma^2\left[\frac{(1+\delta)^{2n}-1}{(1+\delta)^2-1}\right].$$

If $\delta > 0$, then the crack accelerates while if $\delta < 0$ then the crack decelerates.

Case A: $\delta > 0$.

Let $\frac{1}{\beta} = \ln(1+\delta) > 0$. Then

$$F_N(n) = \Phi\left[\frac{\mu_1(e^{n/\beta}-1)-\omega}{\sigma_1\sqrt{e^{2n/\beta}-1}}\right] = \Phi\left[\frac{\mu_2 e^{n/\beta}-\omega_2}{\sqrt{e^{2n/\beta}-1}}\right]$$

where

$$\mu_1 = \frac{\mu}{\delta}, \quad \sigma_1 = \frac{\sigma}{\sqrt{(1+\delta)^2-1}}, \quad \mu_2 = \frac{\mu_1}{\sigma_1}, \quad \omega_2 = \frac{\mu_1+\omega}{\sigma_1}.$$

Let $$\alpha = \frac{1}{\sqrt{\mu_2\omega_2}}, \quad \gamma = \frac{\omega_2}{\mu_2} = 1 + \frac{\omega\delta}{\mu} \doteq 1.$$

Then

$$F_N(n) = \Phi\left[\frac{\frac{1}{\alpha}\left(\sqrt{\frac{e^{n/\beta}}{\gamma}} - \sqrt{\frac{\gamma}{e^{n/\beta}}}\right)}{\sqrt{(e^{n/\beta} - e^{-n/\beta})}}\right] .$$

Hence we have

$$\xi_2(t) = \frac{\xi\left(\frac{e^t}{\gamma}\right)}{[\xi(e^{2t})]^{\frac{1}{2}}} \qquad \text{for} \quad t > 0 .$$

Again we note that

$$\lim_{t \to 0} \xi_2(t) = -\infty, \quad \lim_{t \to \infty} \xi_2(t) = \frac{1}{\sqrt{\gamma}}$$

preventing a reciprocal relationship.

Case B: $\delta < 0$.

Let $\frac{1}{\beta} = -\ln(1+\delta) > 0.$ Then

$$P[N \le n] = \Phi\left[\frac{\mu_1(1-e^{-n/\beta}) - \omega}{\sigma_1\sqrt{1-e^{-2n/\beta}}}\right]$$

where

$$\mu_1 = \frac{\mu}{-\delta} , \quad \sigma_1 = \sigma/\sqrt{1-(1+\delta)^2} .$$

Let

$$\mu_2 = \frac{\mu_1}{\sigma_1} , \quad \omega_2 = \frac{\mu_1-\omega}{\sigma_1} ,$$

and we obtain

$$P[N \le n] = \Phi\left[\frac{\omega_2 e^{n/\beta} - \mu_2}{\sqrt{e^{2n/\beta} - 1}}\right]$$

which is exactly the same model as in the preceding case except that ω_2 and μ_2 are interchanged. Thus the models for crack acceleration and deceleration lie in the same family.

These arguments demonstrate that it may be naive to suppose that a more realistic theory can be formulated without making use of knowledge from other fields, besides mathematical statistics, in the construction of distributional models for fatigue.

SECTION III

THE RELATION BETWEEN GROWTH RATE AND THE DISTRIBUTION OF TIME UNTIL CRITICAL SIZE

Consider a fatigue crack developing in a given structural component due to the repeated impositions of a loading spectrum through its service usage. The crack growth is believed to be divided into several distinct phases, each one of which is governed by a different physical mechanism. Neither the physical-metallurgical basis for this supposition nor the nature of the mechanisms involved will be discussed here. In this regard, see [7] and [8]. Instead, an appropriate stochastic model will be postulated for the behavior of the crack in each phase, each of which constitutes a large source of variation in the observed fatigue life.

The first phase ends when the crack reaches a predetermined detectable size. This is called the initiation phase. We label this minimum detectable size ω_0, which is a function of both the material and the inspection procedure. The time of initiation will vary from component to component even though they are nominally identical structures subjected to the same service loads. Let T_0 be the time when the crack reaches the specified initiation size ω_0. We assume that T_0 is a non-negative random variable, across the population of components, with a distribution F_0 which is unspecified at present: $T_0 \sim F_0$. We do not exclude the possibility that T_0 has a positive probability of being zero. This would correspond with the situation when the specimen was initially flawed.

During the second phase of crack growth the crack grows at a rate which is virtually constant for each component and loading spectrum examined

but the rate will be somewhat different for different specimens even if the same loading spectrum is imposed. Let $s(t)$ denote the observed crack length at any time $t \geq 0$. Given that the crack was initiated at time t_0, our assumption is that for a time thereafter the growth rate is constant, namely

$$s'(t) = u \qquad \text{for } t \geq t_0 .$$

Here the slope u is determined by the material, the work cycle and the specimen itself. It now follows that the observed crack length would be given by

$$s(t) = u(t-t_0) + \omega_0 \qquad \text{for } t \geq t_0 .$$

The constant growth rate of the crack observed during the second phase of behavior varies from one metallic specimen to another and thus the stochastic nature of the crack follows. The population of crack length of these metallic specimens can be described by a stochastic process, given the crack initiation occurred at time t_0; to wit,

$$S(t) = U(t-t_0) + \omega_0 \qquad \text{for } t_0 \leq t \leq t_1 \tag{3}$$

where $U > 0$ is a random variable with a distribution yet to be specified across the population of specimens.

Because of the multitudinous influences governing each incremental crack extension the crack length at a given time can be regarded as a Gaussian Stochastic Process; namely,

$$S(t) \sim \mathfrak{N}(\mu_t, \sigma_t^2) \qquad \text{for each } t > t_0. \tag{4}$$

Moreover, for $t > t_0$, we have

$$\sigma_t = \sigma_0 \sqrt{t-t_0} \qquad \text{and} \qquad \mu_t = m_1(t-t_0) + \omega_0 \ .$$

From equation (4) it follows that the probability that at time $t > t_0$ the crack does not exceed a specified length $\omega > \omega_0$ is

$$P[S(t) > \omega | T_0 = t_0] = \Phi\left(\frac{\mu_t - \omega}{\sigma_t}\right) \qquad \text{for} \quad t > t_0 \ . \tag{5}$$

Here the constant m_1 is the expected rate of crack growth as determined by the loading spectrum and the material. The time scale is chosen so that each application of the spectrum occurs in unit time. We do not regard it as an unknown parameter to be estimated statistically. It should be calculated from the loading spectrum and the material by the use of a formula such as the one given in [9].

Let T_1 be the time at which the crack first reaches the critical propagation length ω_1 and stable, linear growth ends and the third phase of growth begins. During this phase a differential equation following the laws of fracture mechanics is presumed to describe the behavior. The critical length ω_1 is again a constant calculated from the work load and the material.

Thus $$T_1 = \inf\{t: \ S(t) \geq \omega_1\} \ .$$

Define

$$\beta_1 = \frac{\omega_1 - \omega_0}{m_1} \ , \qquad \alpha_1 = \frac{\sigma_0}{\sqrt{m_1(\omega_1-\omega_0)}} \ .$$

Then from equation (5) it follows that for $t > t_0$

$$P[T_1 \le t | T_0 = t_0] = P[S(t) \ge \omega_1 | T_0 = t_0]$$

$$= \Phi\left[\frac{\mu_t - \omega_1}{\sigma_t}\right] = \Phi\left[\frac{1}{\alpha_1}\, \xi\left(\frac{t-t_0}{\beta_1}\right)\right]$$

where

$$\xi(x) = \sqrt{x} - \frac{1}{\sqrt{x}} \qquad \text{for } x > 0 .$$

Thus we see that the distribution of T_1 is ξ-normal with characteristic value β_1 and shape parameter α_1 .

But by suitable rearrangement

$$P[T_1 \le t | T_0 = t_0] = P[U \ge (\omega_1 - \omega_0)/(t-t_0)]$$

and it follows, since $\xi(1/x) = -\xi(x)$, that

$$P[U \le u] = \Phi\left[\frac{1}{\alpha_1}\, \xi(u/m_1)\right] \qquad u > 0 \cdot$$

Thus the distribution of the growth rate (slope) U of the crack during its stable growth is also ξ-normal with characteristic value m_1 and shape parameter α_1 . This reciprocal property of the ξ-normal (Birnbaum-Saunders) distribution has been discussed in [6] and [10].

We summarize this discussion with the following

Theorem 1: Following crack initiation of size ω_0 at given time t_0 , if the crack length $S(t)$ for time $t > t_0$ is a Gaussian Stochastic Process

$$S(t) \sim \mathfrak{N}(\mu_t, \sigma_t^2)$$

where

$$\sigma_t = \sigma_0 \sqrt{t-t_0} \quad , \quad \mu_t = m_1(t-t_0) + \omega_0 \qquad \text{for } t > t_0$$

and each crack progresses at a uniform rate until a critical propagation size ω_1 is reached so that

$$S(t) = U(t-t_0) + \omega_0 \qquad \text{for} \quad t > t_0$$

then the random rate of progress U is a ξ-normal with characteristic value m_1 and shape parameter α_1,

$$U \sim \mathfrak{N}_\xi(\alpha_1, m_1) \; .$$

Moreover the time T_1, following initiation, until the critical size ω_1 is reached is also ξ-normal with characteristic value β_1 and shape parameter α_1,

$$T_1 \sim \mathfrak{N}_\xi(\alpha_1, \beta_1)$$

where

$$\alpha_1 = \frac{\sigma_0}{m_1\sqrt{\beta_1}} \; , \quad \beta_1 = \frac{\omega_1-\omega_0}{m_1} \; .$$

SECTION IV

THE DISTRIBUTION OF TIME UNTIL FATIGUE FAILURE

During the third phase of fatigue crack growth the crack is assumed to be extended as a function of a propagation factor. This propagation factor is a function of time which has been taken as the solution of a differential equation. This solution was derived from an assumption taken from fracture mechanics: namely that the rate of propagation of the crack is a linear function of the square root of the crack length, see [11].

For given values of $a,b > 0$, the propagation factor is of the form

$$p(t: a,b) = \begin{cases} e^{at} & \text{if } b = 0 \\ (1+abt)^{1/b} & \text{if } b \neq 0 . \end{cases}$$

If we suppose that b is fixed for a particular geometry and material then the controlling constant a will have a distribution across specimens drawn at random from the population of such components.

It is then clear that the time spent in the third phase, say T_3, is given by

$$T_3 = \begin{cases} \frac{1}{A} \ln(\omega_3-\omega_2) & \text{if } b = 0 \\ \frac{1}{A}\left[\frac{(\omega_3-\omega_2)^b-1}{b}\right] & \text{if } b \neq 0 . \end{cases}$$

Thus if we assume A is ξ-normal then it would follow that T_3 is ξ-normal as well.

Hence the time until fatigue can be represented as being the sum of three random quantities: the time until crack initiation, the time between initiation and critical size and the time between critical size and ultimate size.

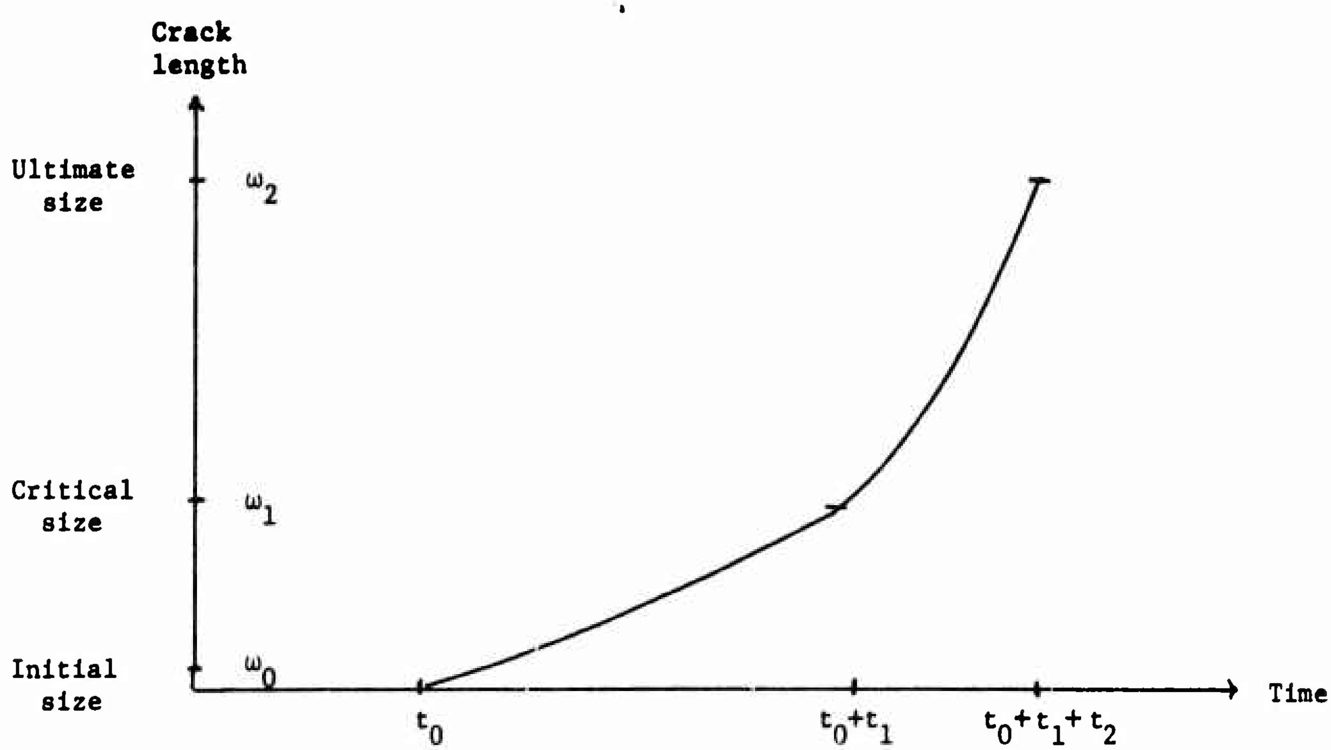

Figure 1. Schematic illustrating separate stages of crack growth.

The relative proportion of time which a crack spends in the three various phases can not be determined in practice. However, there are instances where the crack initiation occurs very late and others where it was suspected to have been initially flawed. It is also thought that the time spent in the third phase is relatively short compared with the other two.

Each of these distinct phases has been studied in the laboratory not only for different materials under different environmental conditions, but different geometrical configurations. Despite this extensive effort there is still so much variability in material, workmanship or service exposure that no calculation of fatigue reliability based upon a synthesis of these theories is universally accepted prior to service with the same assurance that the final fatigue test carries.

Now we come to a critical point in the development. Suppose that we are interested in the convolution of non-negative random variables T_i for

$i = 1,2,\ldots,k$ which are independent but not necessarily identically distributed by one of the ξ-normal distributions. In particular we think of $T_1 + T_2 + T_3$ where

T_1 is the time from the introduction of service until crack inititation;

T_2 is the time between crack initiation and critical crack length;

T_3 is the time following critical crack size until crack growth terminates.

Here crack growth termination may be at component fracture or when the crack reaches a "crack stopper" which is located in certain components.

We ask, what is the distribution of the total time until crack termination?

Here and subsequently, iff means if and only if.

Recall that T is ξ-normal iff for some $\alpha,\beta > 0$

$$Z = \frac{1}{\alpha}\,\xi(T/\beta) \sim \mathfrak{N}(0,1) ,$$

where ξ is a monotone increasing map of $(0,\infty)$ onto $(-\infty,\infty)$ such that

$$\xi(x) = -\xi(1/x) \quad \text{for} \quad x > 0 .$$

This means that $\psi = \xi^{-1}$ must satisfy the relation

$$\psi(-z) = \frac{1}{\psi(z)} \quad \text{for} \quad -\infty < z < \infty \quad \text{and} \quad \psi(0) = 1 . \tag{6}$$

We contend the following: If α_i are sufficiently small and z_i are standard normal variates then

$$\Sigma\, \beta_i\psi_i(\alpha_i z_i) \doteq (\Sigma\, \beta_i)\psi(\Sigma\, \alpha_i z_i) \tag{7}$$

for some function ψ which satisfies (6) as do ψ_i .

Note that for α_i sufficiently small we can certainly make the approximation given in (2) using the first few terms of the Taylor's series approximation. We now note that

$$\sum \beta_i \psi_i(\alpha_i z_i) = \sum \frac{\beta_i}{\psi_i(-\alpha_i z_i)} \doteq (\Sigma \beta_i)^2 [\Sigma \beta_i \psi_i(-\alpha_i z_i)]^{-1} . \quad (8)$$

The last approximation, actually an inequality, follows from the relationship between arithmetic and harmonic means and is exact when all but one of the β_i's are zero. Thus by applying the same argument as before, namely that (7) must hold with z_i replaced by $-z_i$, and then assuming (8) is exact we have

$$\Sigma \beta_i \psi_i(\alpha_i z_i) = \frac{\Sigma \beta_i}{\psi(-\Sigma\alpha_i z_i)}$$

and hence

$$\psi(\Sigma\alpha_i z_i) = \frac{1}{\psi(-\Sigma\alpha_i z_i)} .$$

We now claim that if T_i are ξ_i-normal, then as a mathematical approximation

$$T = T_1 + T_2 + T_3 \quad \text{is also } \xi\text{-normal}$$

but the ξ is different from ξ_i for $i = 1,2,3$. Moreover, we claim that as an empirical fact this approximation is supported by the evidence which has been accumulated on the total time of service from introduction until fatigue fracture. Moreover we feel that ξ-normal variates, which are close to log-normal in construction, are sufficiently descriptive of the fatigue phenomenon that their investigation is of practical importance.

Thus we proceed with the subsequent examination of the problems of estimation which arise in this connection.

The final fatigue test, which can be completed on only a few specimens, because of the time and expense, confounds the effect of each separate phase of the fatigue crack growth within the observation of the total life.

Since the number of complete specimens which can be used for observations of the fatigue life is so small an extremely efficient theory of estimation is needed which is specifically tailored to the data which can be obtained, both in quantity and form.

SECTION V

THE CLASS OF DISTRIBUTIONS

Let Ξ be a class of real valued functions, concave monotone increasing and mapping the positive real line onto the real line such that if $\xi \epsilon \Xi$ then it satisfies the functional equation

$$\xi(t) = -\xi(1/t) \quad \text{for} \quad t > 0. \tag{9}$$

If T is a non-negative random variable such that there exist constants $\alpha, \beta > 0$ and $\xi \epsilon \Xi$ for which

$$\frac{1}{\alpha} \xi(T/\beta) \sim \mathfrak{N}(0,1) \tag{10}$$

we say that T is *ξ-normal;* see [10]. This nomenclature is adopted by analogy from that of the log-normal distribution.

Thus if T is ξ-normal with parameters $\alpha, \beta > 0$ and has distribution F then

$$F(t) = \Phi\left[\frac{1}{\alpha} \xi(t/\beta)\right] \quad \text{for} \quad t > 0$$

where Φ is the standard normal distribution given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt.$$

Virtually all the collections of fatigue life data have been recorded in terms of the logarithm of the number of cycles until failure. Moreover these data have been plotted on log-normal probability graph paper. To facilitate the comparison of these new families of life distributions with plotted data currently available, we shall consider the log-life variate $X = \ell nT$ and we take its distribution G to be of the form

$$G(x) = \Phi\left[\frac{1}{\alpha}\,\omega(x-\mu)\right] \qquad -\infty < x < \infty \tag{11}$$

where

$$\mu = \ell n\beta, \quad \omega(x) = \xi(e^x) \quad \text{for all } x.$$

We now consider the class of functions Ω defined as follows: For a real valued function ω, mapping the real line onto itself, which is twice differentiable we say;

$$\omega \in \Omega \text{ iff } \omega' \geq 0, \; \omega' \geq \omega'' \text{ and } \omega \text{ is odd.} \tag{12}$$

Let us examine the closure properties of Ω.

We state without proof:

Theorem 2: If $\omega_1, \omega_2 \in \Omega$ then $\omega_1 + \omega_2 \in \Omega$. Moreover if $\omega \in \Omega$ and $a > 0$, $0 < b < 1$ then both $\omega_1, \omega_2 \in \Omega$ where, for $-\infty < x < \infty$, $\omega_1(x) = a\omega(x)$ and $\omega_2(x) = \omega(bx)$.

As a consequence, for any given $\omega \in \Omega$ we can generate a three-dimensional parametric family of distributions of the log-life, namely,

$$G(x:\alpha,\gamma,\mu) = \Phi\left[\frac{1}{\alpha}\,\omega\left(\frac{x-\mu}{\gamma}\right)\right] \qquad -\infty < x < \infty \tag{13}$$

where

$$\alpha > 0, \quad \gamma > 0, \quad -\infty < \mu < \infty .$$

In this notation μ is the *location* parameter, γ is the *scale* parameter and α is the *flexure* parameter. The flexure and scale together control the shape.

Examples of such functions ω which correspond to known parametric families are:

$$\omega_1(x) = \sinh x \qquad -\infty < x < \infty \tag{13}$$

and

$$\omega_2(x) = \Phi^{-1}[F(x)] \qquad -\infty < x < \infty$$

where

$$F(x) = \frac{1-\operatorname{sgn} x}{2} + (\operatorname{sgn} x)\exp\{-(\ln 2)e^{-|x|}\}.$$

Here ω_1 corresponds to the distributions of Birnbaum-Saunders in [6] and ω_2 corresponds to the symmetric Weibull distributions of [12].

There are several ways of constructing functions in Ω. One is as follows: Given any real valued function f, mapping the real line onto itself, such that

$$f' \geq f'' > 0 \quad \text{define} \quad \omega(x) = f(x) - f(-x).$$

A second method follows from Theorem 2. For example, if we have constants $a_i > 0$, $1 > b_i > 0$ for $i = 1,\dots,n$ then $\omega\epsilon\Omega$ where

$$\omega(x) = \sum_1^n 2a_i \sinh(b_i x) \qquad -\infty < x < \infty .$$

Now we state a result on closure under inverse transformations in

Theorem 3: If $\omega\epsilon\Omega$ and $(\omega')^2 \geq -\omega''$ then $\omega^{-1}\epsilon\Omega$.

Proof: Clearly

$$(\omega^{-1})' = 1/\omega'(\omega^{-1}) \geq 0 \quad \text{since} \quad \omega' \geq 0,$$

and ω odd implies ω^{-1} is odd. We must only check that $(\omega')'' \leq (\omega^{-1})'$.

By definition

$$(\omega^{-1})'' = \frac{-[\omega'(\omega^{-1})]'}{[\omega'(\omega^{-1})]^2} = \frac{-\omega''(\omega^{-1})}{[\omega'(\omega^{-1})]^3} \quad .$$

But

$$\frac{-\omega''(\omega^{-1})}{[\omega'(\omega^{-1})]^3} \leq \frac{1}{\omega'(\omega^{-1})}$$

is equivalent with

$$-\omega''(\omega^{-1}) \leq [\omega'(\omega^{-1})]^2 \quad \text{iff} \quad -\omega'' \leq (\omega')^2 \ . \ ||$$

Thus there are inverses of certain elements of Ω which are themselves included in Ω. Using the notation of (13) we have

Theorem 4: $\omega_1^{-1} \varepsilon \Omega$ where $\omega_1(x) = \sinh x$, and we define

$$\omega_3(x) = \omega_1^{-1}(x) = \ln(x + \sqrt{x^2 + 1}) \quad . \tag{14}$$

Proof: By definition

$$\omega_1 = \sinh, \quad \omega_1' = \cosh, \quad \omega_1'' = \omega_1.$$

We must, by theorem 2, check that

$$-\sinh x \leq (\cosh x)^2 \quad \text{for all} \quad x \ , \quad -\infty < x < \infty \ .$$

If we let x be replaced by $-x$, this inequality is equivalent with

$$-\sinh(-x) \leq [\cosh(-x)]^2.$$

But by known properties of the hyperbolic functions this is equivalent with

$$\sinh x \leq (\cosh x)^2 \qquad -\infty < x < \infty \ .$$

From the definitions we must show that

$$(e^{x}-e^{-x})/2 \leq (e^{2x}+2+e^{-2x})/4 \ ,$$

which is equivalent with

$$0 \leq e^{2x} - 2e^{x} + 2 + 2e^{-x} + e^{-2x} = (e^{x}-1)^2 + (1+e^{-x})^2 \ . \ ||$$

Let us illustrate the typical behavior of one such family and its reciprocal. Consider the two three-parameter families induced by (13) for $\alpha > 0$, $\gamma > 0$, $-\infty < \mu < \infty$ which are generated by ω_1 and ω_3.

$$\omega(x) = \frac{1}{\alpha} \sinh\left(\frac{x-\mu}{\gamma}\right)$$

$$\omega^*(x) = \frac{1}{\alpha} \sinh^{-1}\left(\frac{x-\mu}{\gamma}\right) = \frac{1}{\alpha} \ln\left[\frac{x-\mu + \sqrt{(x-\mu)^2+\gamma^2}}{\gamma}\right]$$

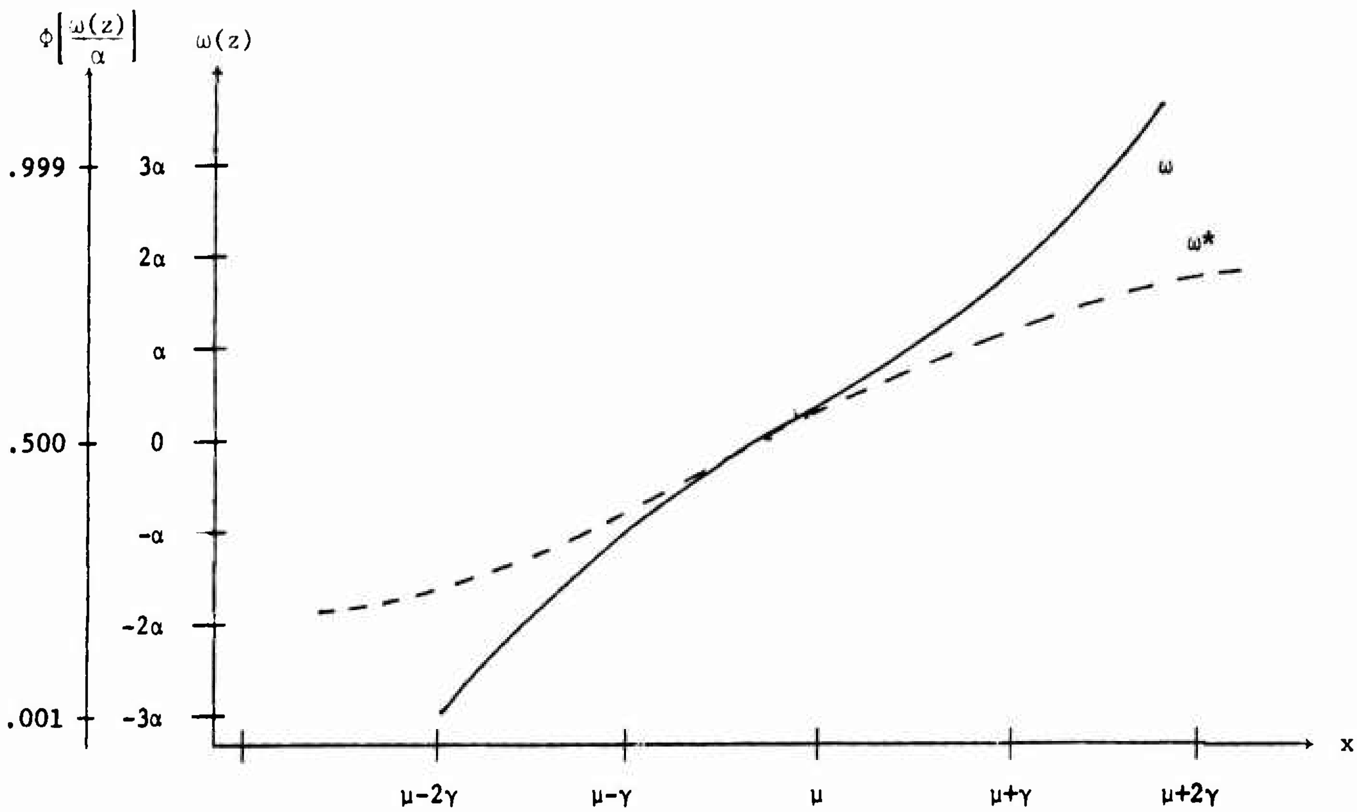

Figure 2. Specific transformations; $x = \mu+z$ plotted against $\omega(z)$ and $\Phi\left[\frac{\omega(z)}{\alpha}\right]$.

SECTION VI

THE ESTIMATION OF PARAMETERS BY MAXIMUM LIKELIHOOD

Let us suppose that we have a complete sample of failure data $(x_1,\ldots,x_n)$ from the distribution

$$G(x) = \Phi\left[\frac{1}{\alpha}\,\omega\left(\frac{x-\mu}{\gamma}\right)\right] \qquad -\infty < x < \infty$$

where ω is known but the parameters α,γ,μ are unknown. The density of each observation is

$$g(x) = \phi\left[\frac{1}{\alpha}\,\omega\left(\frac{x-\mu}{\gamma}\right)\right]\frac{1}{\alpha\gamma}\quad \omega'\left(\frac{x-\mu}{\gamma}\right)$$

and thus

$$\ell n\, g(x) = \frac{-\ell n(2\pi)}{2} - \frac{1}{2\alpha^2}\,\omega^2\left(\frac{x-\mu}{\gamma}\right) - \ell n(\alpha\gamma) + \ell n\,\omega'\left(\frac{x-\mu}{\gamma}\right).$$

The log-likelihood then is $\sum_{1}^{n} \ell n\, g(x_i)$

which, except for a constant independent of the parameters, is

$$L = \sum_{1}^{n}\left\{-\ell n(\alpha\gamma) - \frac{1}{2\alpha^2}\,\omega^2\left(\frac{x_i-\mu}{\gamma}\right) + \ell n\,\omega'\left(\frac{x_i-\mu}{\gamma}\right)\right\}. \tag{15}$$

In order to maximize the likelihood we consider

$$\frac{\partial L}{\partial \mu} = \sum_{1}^{n}\left\{\frac{1}{\alpha^2\gamma}\omega\left(\frac{x_i-\mu}{\gamma}\right)\ \omega'\left(\frac{x_i-\mu}{\gamma}\right) - \frac{1}{2}\,\frac{\omega''\left(\frac{x_i-\mu}{\gamma}\right)}{\omega'\left(\frac{xi-\mu}{\gamma}\right)}\right\} \tag{16}$$

$$\frac{\partial L}{\partial \gamma} = \sum_{1}^{n}\left\{-\frac{1}{\gamma} + \frac{1}{\alpha^2}\omega\left(\frac{x_i-\mu}{\gamma}\right)\omega'\left(\frac{x_i-\mu}{\gamma}\right)\left(\frac{x_i-\mu}{\gamma^2}\right) - \frac{\omega''\left(\frac{x_i-\mu}{\gamma}\right)}{\omega'\left(\frac{x_i-\mu}{\gamma}\right)}\left(\frac{x_i-\mu}{\gamma^2}\right)\right\} \tag{17}$$

$$\frac{\partial L}{\partial \alpha} = \sum_{1}^{n}\left\{-\frac{1}{\alpha} + \frac{\omega^2\left(\frac{x_i-\mu}{\gamma}\right)}{\alpha^3}\right\}. \tag{18}$$

Now the joint solution of $\frac{\partial L}{\partial \mu} = 0$, $\frac{\partial L}{\partial \gamma} = 0$, $\frac{\partial L}{\partial \alpha} = 0$ will yield the maximum likelihood estimators $\hat{\alpha}, \hat{\gamma}, \hat{\mu}$. We thus seek the simultaneous solution to the following equations in the variables (α, γ, μ):

$$\frac{1}{\alpha^2} \sum_1^n \omega\left(\frac{x_i-\mu}{\gamma}\right)\omega'\left(\frac{x_i-\mu}{\gamma}\right) = \sum_1^n \frac{\omega''\left(\frac{x_i-\mu}{\gamma}\right)}{\omega'\left(\frac{x_i-\mu}{\gamma}\right)} , \tag{19}$$

$$\frac{1}{\alpha^2}\frac{1}{n} \sum_1^n \left(\frac{x_i-\mu}{\gamma}\right) \omega'\left(\frac{x_i-\mu}{\gamma}\right)\omega\left(\frac{x_i-\mu}{\gamma}\right) - \frac{1}{n} \sum_1^n \left(\frac{x_i-\mu}{\gamma}\right) \frac{\omega''\left(\frac{x_i-\mu}{\gamma}\right)}{\omega'\left(\frac{x_i-\mu}{\gamma}\right)} = 1 , \tag{20}$$

$$\alpha^2 = \frac{1}{n} \sum_1^n \omega^2\left(\frac{x_i-\mu}{\gamma}\right) . \tag{21}$$

As a notational convenience, let us define as implicit functions of μ and γ the values

$$y_i = \frac{x_i-\mu}{\gamma} \qquad i = 1,\dots,n$$

and introduce, for any function f, the averaging operator

$$\langle f(y_i)\rangle = \frac{1}{n} \sum_1^n f(y_i) .$$

We define as functions of (μ,γ):

$$\bar{H} = \frac{\langle P(y_i)\rangle}{\langle \omega^2(y_i)\rangle} - \langle R(y_i)\rangle ,$$

and

$$H^* = \frac{\langle y_i P(y_i)\rangle}{\langle \omega^2(y_i)\rangle} - \langle y_i R(y_i)\rangle - 1 ,$$

where we have the ratio and product functions defined when both R and P are odd, by

$$R = \omega''/\omega', \quad P = \omega\,\omega' . \tag{22}$$

Let us suppose throughout the following discussion that we have several samples from different populations. Suppose we have m such samples

$$x_{1,j},\ldots,x_{n_j,j} \quad \text{for} \quad j = 1,\ldots,m.$$

where n_j observations are taken from the j^{th} population with parameters $(\alpha_j,\gamma_j,\mu_j)$ for $j = 1,\ldots,m$. Let us compute the two sample statistics

$$\bar{x}_j = \frac{1}{n_j}\sum_{i=1}^{n_j} x_{ij}, \quad s_j^2 = \frac{1}{n_j-1}\sum_{i=1}^{n_j}(x_{ij} - \bar{x}_j)^2$$

of

$$X_{ij} \sim I(\alpha_j,\gamma_j,\mu_j) \qquad \text{for} \quad i = 1,\ldots,n_j;$$

then

$$Z_{ij} = \frac{1}{\alpha_j}\,\omega\left(\frac{X_{ij}-\mu_j}{\gamma_j}\right) \qquad i = 1,\ldots,n_j; \quad j = 1,\ldots,m$$

are independent identically distributed standard normal variates. Since ω^{-1} is odd we see

$$E\left(\frac{X_{ij}-\mu_j}{\gamma_j}\right) = E\ \omega^{-1}(\alpha_j Z_{ij}) = 0$$

and hence

$$E\ X_{ij} = \mu_j$$

and thus $\bar{X}_j$ is an unbiased estimate of μ_j. But note also that

$$E\left[\omega^{-1}(\alpha_j Z_{ij})\right]^2 = E\left(\frac{X_{ij}-\mu_j}{\gamma_j}\right)^2 .$$

If we define

$$\sigma_j^2 = \mathrm{var}(X_{ij})$$

we see

$$\sigma_j^2 = \gamma_j^2 \, E \, \omega^{-1}(\alpha_j Z_{ij})^2$$

and we know that an unbiased estimate of σ_j^2 is the sample standard deviation s_j^2.

Thus we see that this family of distributions separates the variance into the product of two factors. The first factor is the scale parameter while the second factor, say $B(\alpha)$, is determined by the flexure parameter α namely

$$B(\alpha) = E \, \omega^{-1}(\alpha Z)^2 = \int_{-\infty}^{\infty} \omega^{-1}(y)^2 d\Phi\left(\frac{y}{\alpha}\right) .$$

Thus the equation, valid for large samples, of

$$s_j^2 = \gamma_j^2 \, B(\alpha_j)$$

forces us to utilize an independent method for estimating α_j.

We now prove the important

Theorem 5: The maximum likelihood estimator of α_j, defined by

$$\hat{\alpha}_j = \left[\frac{1}{n_j}\sum_{i=1}^{n_j} \omega^2\left(\frac{x_{ij}-\hat{\mu}_j}{\hat{\gamma}_j}\right)\right]^{\frac{1}{2}} , \tag{23}$$

has a distribution independent of μ_j and γ_j.

Proof: Let

$$y_{ij} = \frac{x_{ij}-\mu_j}{\gamma_j} \qquad \text{for } i = 1,\ldots,n_j$$

be standardized variables independent of μ_j and γ_j and

define

$$U_j = \frac{\hat{\mu}_j - \mu_j}{\gamma_j} \qquad V_j = \hat{\gamma}_j/\gamma_j$$

as the normalized maximum likelihood estimators. We now write

$$T_{ij} = \frac{X_{ij} - \hat{\mu}_j}{\hat{\gamma}_j} = \left(\frac{X_{ij} - \mu_j}{\gamma_j} + \frac{\mu_j - \hat{\mu}_j}{\gamma_j}\right)\Big/\left(\hat{\gamma}_j/\gamma_j\right) = \frac{Y_{ij} - U_j}{V_j} \tag{24}$$

and we note that for each j the m.ℓ.e.'s, namely $\hat{\mu}_j$ and $\hat{\gamma}_j$, satisfy the two equations which we can write as

$$\langle P(T_{ij})\rangle = \langle R(T_{ij})\rangle\langle \omega^2(T_{ij})\rangle$$

$$\langle T_{ij}P(T_{ij})\rangle = \left[\langle T_{ij}R(T_{ij})\rangle + 1\right]\langle \omega^2(T_{ij})\rangle ,$$

making use of the product and ratio functions P and R as defined in (22) . The averaging operator $\langle\cdot\rangle$ is taken only over the subscript i. Since the Y_{ij} for $i = 1,\ldots,n_j$ have distributions independent of μ_j and γ_j , it follows *a fortiori* that so must $\hat{\alpha}_j$.||

Presume that we have computed the maximum likelihood estimates previously defined for m different groups of data as

$$\hat{\alpha}_j,\ \hat{\gamma}_j,\ \hat{\mu}_j \qquad \text{for } j = 1,\ldots,m.$$

Under the null hypotheses that all the α_j's are the same, i.e.,

$$H_0:\ \alpha_1 = \alpha_2 = \ldots = \alpha_m ,$$

we might take a grand average

$$\tilde{\alpha}^2 = \sum_{j=1}^{m} n_j\,\hat{\alpha}_j^2 \Big/ \sum_{j=1}^{m} n_j$$

as the combined estimate.

We know from general statistical theory on the consistency of the maximum likelihood estimators that $\hat{\alpha}_j^2 \to \alpha_j^2$ in probability as $n_j \to \infty$. Unfortunately we have rather small sample sizes n_j for $j = 1,\ldots,m$ but the number of groups m may be quite large. Thus if it were possible we would determine bias factors $b(n)$, as a function of the sample size $n = 1,2,\ldots,$ such that

$$E\,\hat{\alpha}_j^2 = \alpha_j^2\, b(n_j)$$

from which we could define unbiased estimates and have consistency as $m \to \infty$ as well. However from (3.1) we have

$$E\,\hat{\alpha}_j^2 = E\,\omega^2(T_{ij})$$

where T_{ij} is defined in (24). Thus one cannot guarantee that α_j can be factored out leaving b a function only of the sample size. We conjecture that, in fact, it is the case that b will also be a function of α.

We now turn to the problem of estimating the common α and the related problem of testing the hypotheses H_0 from the point of view of general statistical theory. Let us suppose that H_0 is true so that

$$X_{ij} \sim I(\alpha,\gamma_j,\mu_j) \qquad \text{for} \quad i = 1,\ldots,n_j\ .$$

The joint likelihood is the sum of likelihoods for each of the m sets of data, each one of which is similar to the one before, namely

$$L = \sum_{j=1}^{m} L_j$$

where by comparison with (15) we see

$$L_j = \sum_{i=1}^{n_j} \left\{ -\ln(\alpha\gamma_j) - \frac{1}{2\alpha^2}\,\omega^2\!\left(\frac{x_{ij}-\mu_j}{\gamma_j}\right) + \ln \omega'\, \frac{x_{ij}-\mu_j}{\gamma_j} \right\}.$$

By comparison with (16) we have for $j = 1,\ldots,m$

$$\frac{\partial L}{\partial \mu_j} = \sum_{i=1}^{n_j} \left\{ \frac{1}{\alpha^2 \gamma_j} P\left(\frac{x_{ij}-\mu_j}{\gamma_j}\right) - \frac{1}{\gamma_j} R\left(\frac{x_{ij}-\mu_j}{\gamma_j}\right) \right\}$$

where we have made use of P and R as defined in (22). Similarly, we have from (17) for $j = 1,\ldots,m$

$$\frac{\partial L}{\partial \gamma_j} = \sum_{i=1}^{n_j} \left\{ -\frac{1}{\gamma_j} + \frac{1}{\alpha^2 \gamma_j}\left(\frac{x_{ij}-\mu_j}{\gamma_j}\right) P\left(\frac{x_{ij}-\mu_j}{\gamma_j}\right) - \frac{1}{\gamma_j}\left(\frac{x_{ij}-\mu_j}{\gamma_j}\right) R\left(\frac{x_{ij}-\mu_j}{\gamma_j}\right) \right\}$$

and from Eq. (18)

$$\frac{\partial L}{\partial \alpha} = \sum_{j=1}^{m} \sum_{i=1}^{n_j} \left[-\frac{1}{\alpha} + \frac{1}{\alpha^3} \omega^2\left(\frac{x_{ij}-\mu_j}{\gamma_j}\right) \right].$$

The method of obtaining the maximum likelihood estimators then is as follows: We must solve for the $2m + 1$ variables

$$\alpha, \gamma_1, \ldots \gamma_m, \mu_1, \ldots, \mu_m$$

in the $2m + 1$ equations which we can write, letting $y_{ij} = \dfrac{x_{ij}-\mu_j}{\gamma_j}$,

$$\frac{1}{\alpha^2} \frac{1}{n_j} \sum_{i=1}^{n_j} P(y_{ij}) = \frac{1}{n_j} \sum_{i=1}^{n_j} R(y_{ij}) \qquad j = 1,\ldots,m$$

$$\frac{1}{\alpha^2} \frac{1}{n_j} \sum_{i=1}^{n_j} y_{ij} P(y_{ij}) - \frac{1}{n_j} \sum_{i=1}^{n_j} y_{ij} R(y_{ij}) = 1 \qquad j = 1,\ldots,m,$$

$$\alpha^2 = \sum_{j=1}^{m} \sum_{i=1}^{n_j} \omega^2(y_{ij}) \Big/ \sum_{j=1}^{m} n_j .$$

Let us call the simultaneous solution $\alpha^*, \gamma^*, \ldots, \gamma_m^*, \mu_1^*, \ldots, \mu_m^*$. The actual method of solution we defer until later.

Let

$$\theta = (\alpha_1, \ldots, \alpha_m, \gamma_1, \ldots, \gamma_m, \mu_1, \ldots, \mu_m).$$

Define the parameter space Ω_1 and a subset Ω_0 as follows:

$$\theta \in \Omega_1 \quad \text{iff} \quad \alpha_i > 0, \quad \gamma_i > 0, \quad -\infty < \mu_i < \infty \quad \text{for } i = 1, \ldots, m$$

$$\theta \in \Omega_0 \quad \text{iff} \quad \theta \in \Omega_1 \quad \text{and} \quad \alpha_1 = \alpha_2 = \ldots = \alpha_m.$$

These two sets correspond to the two hypotheses.

$$H_0: \ \theta \in \Omega_0, \qquad H_1: \ \theta \in \Omega_1 .$$

The likelihood ratio is

$$\lambda = \frac{\sup\limits_{\theta \in \Omega_0} \prod\limits_{j=1} \prod\limits_{i=1} g(x_{ij}:\alpha, \gamma_j, \mu_j)}{\sup\limits_{\theta \in \Omega_1} \prod\limits_{j=1} \prod\limits_{i=1} g(x_{ij}:\alpha_j, \gamma_j, \mu_j)} .$$

We know by statistical theory that

$$-2\ln\lambda = -2 \sum_{j=1}^{m} \sum_{i=1}^{n_j} \left[\ln g(x_{ij}:\alpha^*, \gamma_j^*, \mu_j^*) - \ln g(x_{ij}:\hat{\alpha}_j, \hat{\gamma}_j, \hat{\mu}_j) \right]$$

will have, asymptotically, a chi-square distribution with $3m-(2m+1) = m-1$ degrees of freedom. Thus the critical region will be the upper tail of the $\chi^2(m-1)$ distribution.

In order to compute this test statistic, we use (15) and set

$$\hat{y}_{ij} = \frac{x_{ij}-\hat{\mu}_j}{\hat{\gamma}_j}, \quad y_{ij}^* = \frac{x_{ij}-\mu_j^*}{\gamma_j^*}$$

to write

$$-2\ln\lambda = -2 \sum_{j=1}^{m} \sum_{i=1}^{n_j} \Bigg\{ - \ln \alpha^* \gamma_j^* - \frac{1}{2(\alpha^*)^2} \omega^2(y_{ij}^*) + \ln \omega'(y_{ij}^*)$$

$$+ \ln \hat{\alpha}_j \hat{\gamma}_j + \frac{1}{2(\hat{\alpha}_j)^2} \omega^2(\hat{y}_{ij}) - \ln \omega'(\hat{y}_{ij}) \Bigg\} .$$

This is the statistic that must be computed to perform the test.

SECTION VII

THE BEHAVIOR OF THE MAXIMUM LIKELIHOOD FUNCTION IN CERTAIN CASES

We now consider the behavior of the likelihood function when ω is "s-shaped" in its behavior. The typical shape is that of ω^* as graphed in Figure 2. Formally we say that ω is s-shaped if

(i) $1 \geq \frac{\omega(x)}{x} \geq \omega'(x) > 0 \quad , \qquad \omega'(0) = 1$

(ii) $\omega'(x)$ decreases as $|x|$ increases.

Let us consider the joint log-likelihood equation in· lving only the two parameters μ, γ , having previously eliminated α by substitution. We obtain

$$L(\mu,\gamma) = -\ell n(\alpha\gamma) - \frac{1}{2n(\hat{\alpha})^2} \sum_{i=1}^{n} \omega^2\left(\frac{x_i-\mu}{\gamma}\right) + \frac{1}{n}\sum_{i=1}^{n} \ell n\ \omega'\left(\frac{x_i-\mu}{\gamma}\right) \qquad (25)$$

where α is a function of μ, γ given by

$$(\hat{\alpha})^2 = \frac{1}{n}\sum_{i=1}^{n} \omega^2\left(\frac{x_i-\mu}{\gamma}\right) .$$

We find after substitution and simplification that

$$L(\mu,\gamma) = -\frac{1}{2}\ \ell n \left\{ \frac{1}{n} \sum_{1}^{n} \left[\gamma\ \omega\left(\frac{x_i-\mu}{\gamma}\right)\right]^2 \right\} - \frac{1}{2} + \frac{1}{n} \sum_{1}^{n} \ell n\ \omega'\left(\frac{x_i-\mu}{\gamma}\right) . \qquad (26)$$

Let us set

$$y_i = x_i - \mu \qquad \text{for} \quad i = 1,\ldots,n \quad , \quad a = \max_{i=1}^{n} |y_i| \ .$$

Now from property (i) we have for $i = 1,\ldots,n$

$$[\gamma\, \omega(y_i/\gamma)]^2 \geq y_i^2[\omega'(y_i/\gamma)]^2 \geq y_i^2[\omega'(a/\gamma)]^2 \ .$$

Upon substitution we find

$$L(\mu,\gamma) + \frac{1}{2}\,\ell n\left\{\frac{1}{n}\sum_{1}^{n} y^2\right\} + \frac{1}{2} \leq -\,\ell n\ \omega'(a/\gamma) \ .$$

On the other hand we have, by the bound in (i)

$$[\gamma\ \omega(y_i/\gamma)]^2 \ = \ y_i^2\left[\frac{\omega(y_i/\gamma)}{y_i/\gamma}\right]^2 \leq y_i^2 \qquad \text{for} \quad i = 1,\ldots,n$$

and thus

$$L(\mu,\gamma) \geq -\,\frac{1}{2}\,\ell n\left\{\frac{1}{n}\sum_{1}^{n} y_i^2\right\} - \frac{1}{2} + \frac{1}{n}\sum_{i=1}^{n} \ell n\ \omega'(y_i/\gamma) \quad . \tag{27}$$

From (ii) we know that

$$\omega'(y_i/\gamma) \geq \omega'(a/\gamma) \qquad \text{for} \quad i = 1,\ldots,n$$

and thus from (27) we have

$$L(\mu,\gamma) + \ell n \left\{\frac{1}{n}\sum_{1}^{n} y_i^2\right\} + \frac{1}{2} \geq \ell n\ \omega'(a/\gamma) \quad .$$

We can now state

Theorem 6: The log-likelihood, as a function of $\gamma > 0$, is bounded by

$$\ln \omega'(a/\gamma) \le L(\mu,\gamma) + \frac{1}{2} \ln \left\{ \frac{1}{n} \sum_{i=1}^{n} (x_i-\mu)^2 \right\} + \frac{1}{2} \le - \ln \omega'(a/\gamma)$$

where

$$a = \max_{i=1}^{n} |x_i-\mu| \quad .$$

It follows, of course, that in the limit as $\gamma \to \infty$ the log-likelihood approaches a function of μ which has a maximum at the value $\mu = \bar{x}$. Moreover, certain of those ω's chosen for numerical examination show that for fixed μ , the likelihood is an increasing function of $\gamma > 0$ and moreover that the limit is reached very quickly. To the postulated behavior of s-shaped ω's in Ω we add the assumption

(iii) $\zeta_\omega(x) = \dfrac{x\,\omega'(x)}{\omega(x)}$ decreases as $|x|$ increases.

For the remainder of this discussion we fix μ , omit its mention in equation (26), and write for the likelihood simply

$$L(\gamma) = -\frac{1}{2} \ln \left[\frac{1}{n} \sum_{1}^{n} h_i^2(\gamma) \right] - \frac{1}{2} + \frac{1}{n} \sum_{1}^{n} \ln \omega'(y_i/\gamma)$$

where

$$y_i = x_i - \mu \quad \text{for } i = 1,\ldots,n$$

and

$$h_i(\gamma) = \gamma\, \omega(|y_i|/\gamma) \quad .$$

This equality follows since ω is odd and positive for positive argument and therefore $x\omega(x)$ and $\omega'(x)$ are even and positive for all real x .

By adding and subtracting the quantity $\frac{1}{n}\sum_1^n \ell n\, h_i(\gamma)$ and then combining terms, we obtain, in new notation,

$$L(\gamma) = C(\gamma) + D(\gamma) - \frac{1}{2} - \frac{1}{n}\sum_1^n \ell n|y_i| \tag{28}$$

where the functions C and D are defined respectively for $\gamma > 0$ by

$$C(\gamma) = \frac{1}{n}\sum_1^n \ell n\, \zeta(|y_i|/\gamma) \tag{29}$$

and

$$D(\gamma) = \frac{1}{n}\sum_1^n \ell n\, h_i(\gamma) - \frac{1}{2}\ell n\left[\frac{1}{n}\sum_1^n h_i^2(\gamma)\right] . \tag{30}$$

We study their behavior in

Theorem 7: The function D is always negative and decreasing, while C is negative and increasing.

Proof: To see that D is negative we recall Jensen's Inequality, that for any random variable X, if ϕ is convex then

$$\phi(EX) \le E\,\phi(X) .$$

Since $-\frac{1}{2}\ell n\, x$ is convex (and decreasing) we see

$$-\frac{1}{2}\ell n\left(\frac{1}{n}\sum_1^n h_i^2\right) \le -\frac{1}{n}\sum_1^n \ell n\, h_i ,$$

from which we conclude that $D \le 0$. To see that D decreases we show

$$D' = \frac{1}{n}\sum_1^n \frac{h_i'}{h_i} - \left(\sum_1^n h_i h_i' / \sum_1^n h_i^2\right) \le 0 .$$

This is equivalent to

$$\frac{1}{n}\sum a_i b_i \geq \frac{1}{n}\sum a_i \frac{1}{n}\sum b_i \tag{31}$$

where

$$a_i = h_i^2(\gamma) = [\gamma\, \omega(|y_i|/\gamma)]^2 > 0$$

and

$$b_i = \frac{\gamma h_i'(\gamma)}{h_i(\gamma)} = 1 - \zeta_\omega(|y_i|/\gamma) > 0 \ .$$

But the inequality (31) is true provided that

$$(a_i - a_j)(b_i - b_j) \geq 0 \qquad \text{for all} \quad i,j \ . \tag{32}$$

(See Hardy, Littlewood and Polya, [13] p. 43).

Suppose $|y_i| < |y_j|$. Firstly since $\omega^2(x)$ is an even function which increases as $|x|$ increases we have for fixed γ

$$a_i = \gamma^2\, \omega^2(|y_i|/\gamma) < \gamma^2\, \omega^2(|y_j|/\gamma) = a_j \ .$$

Secondly since $\zeta(x)$ is also an even function but one which decreases as $|x|$ increases we see from (iii) and our supposition that

$$b_i = 1 - \zeta_\omega(|y_i|/\gamma) < 1 - \zeta_\omega(|y_j|/\gamma) = b_j \ .$$

Thus we see the sequences are similarly ordered and (32) is true. Hence D' is negative and therefore D is decreasing.

To see that C is negative we note from (iii) that $1 = \zeta_\omega(0) \geq \zeta_\omega(x)$ and since ζ_ω decreases, it follows that $\ln \zeta_\omega(|y_i|/\gamma)$ increases as a function of γ . $||$

We now show that D is virtually constant.

Theorem 8: Let

$$\eta_\omega(y) = \lim_{x \to \infty} \frac{\omega(yx)}{\omega(x)} \quad \text{for } y > 0$$

then

$$D(0+) = -\frac{1}{2n} \sum_{i=1}^{n} \ln \left[\frac{1}{n} \sum_{j=1}^{n} \eta_\omega^2(|y_j/y_i|) \right]$$

and

$$D(\infty) = \frac{1}{n} \sum_{1}^{n} \ln|y_i| - \frac{1}{2} \ln \left[\frac{1}{n} \sum_{j=1}^{n} |y_i|^2 \right] .$$

Moreover

$$D(\infty) \le D(0+) \le 0 .$$

Proof: Note that $h_i(\gamma) = |y_i|\ \phi(|y_i|/\gamma) \quad i = 1,\ldots,n$ and since $\phi(0+) = 1$ we have $\lim_{\gamma \to \infty} h_i(\gamma) = |y_i|$. From this fact and the definition given in (30) we see that $D(\infty)$ is as given. An expression equivalent to (30) is

$$D(\gamma) = \frac{-1}{2n} \sum_{i=1}^{n} \ln \left[\frac{1}{n} \sum_{j=1}^{n} \frac{h_j^2(\gamma)}{h_i^2(\gamma)} \right] .$$

Now by definition:

$$\lim_{\gamma \to 0} [h_j(\gamma)/h_i(\gamma)] = \lim_{x \to \infty} \frac{\omega(|y_j/y_i|x)}{\omega(x)} = \eta_\omega(|y_j/y_i|)$$

and hence the result follows. $||$

We now show that the range of C is quite extensive.

Theorem 9: In all cases $C(\infty) = 0$, while

$$C(0+) = \begin{cases} \ln \delta & \text{if } \delta > 0 \\ \\ -\infty & \text{if } \delta = 0 \end{cases}$$

where

$$\delta = \lim_{x \to \infty} \delta_\omega(x) \qquad 0 \le \delta < 1 .$$

We now show the relationship of the S-shapedness of a function and the behavior of its inverse:

Theorem 10: ω is S-shaped i.e. it satisfies properties (i), (ii) and (iii) iff its inverse $\psi = \omega^{-1}$ satisfies (i)', (ii)', (iii)' where

(i)' $\qquad 1 \le \dfrac{\psi(x)}{x} \le \omega'(x)$,

(ii)' $\qquad \psi'(x)$ increases as $|x|$ increases ,

(iii)' $\qquad \dfrac{x\psi'(x)}{\psi(x)}$ increases as $|x|$ increases.

Proof: First note that $\omega = \psi^{-1}$ and $\omega'(\psi) = \frac{1}{\psi'}$, then let $x = \psi(y)$, which is an order preserving transformation, so that

$$\frac{\omega(x)}{x\ \omega'(x)} = \frac{\omega[\psi(y)]}{\psi(y)\ \omega'[\psi(y)]} = \frac{y\ \psi'(y)}{\psi(y)} .$$

Thus we see the claim is true for (iii)' . The others are checked in a similar way. $||$

Consider the boundary case where

$$\omega(x) = |x|^a \operatorname{sgn} x \qquad -\infty < x < \infty$$

for some $a > 0$. Now ω satisfies property (iii) iff $a < 1$. But it follows that

$$\zeta_\omega(x) = \frac{x\,\omega'(x)}{\omega(x)} \equiv a \, .$$

Consequently from the form of the likelihood given in (28) we see that C is constant. Moreover we check easily that D is constant also. Therefore the likelihood is constant for all $\gamma > 0$. As an illustration of the applicability of Theorem 5 we consider

$$\omega(x) = \ln\left(x + \sqrt{x^2+1}\right) \, , \qquad \omega'(x) = (1+x^2)^{-\frac{1}{2}} \, ,$$

$$\omega''(x) = \frac{-x}{(1+x^2)^{\frac{3}{2}}} \, , \qquad \omega'''(x) = \frac{(1+x^2)^{\frac{1}{2}}(-1+2x^2)}{(1+x^2)^3} \, . \tag{33}$$

Therefore we see ω is increasing and concave for $x > 0$ since $\omega' > 0$ and ω' is decreasing as $|x|$ increases. To check the remaining condition we examine the inverse

$$\psi(x) = \omega^{-1}(x) = \sinh(x) \, .$$

Then

$$\zeta_\psi(u) = \frac{\sinh u}{u \cosh u} = \frac{\tanh u}{u} \, .$$

Thus we see

$\zeta'_\psi(u) \le 0$ iff $u \operatorname{sech}^2(u)$ iff $2u \le \sinh(2u)$ which is clearly

true since for all real u

$$\sinh(u) = u + \frac{u^3}{3!} + \frac{u^5}{5!} + \dots \quad .$$

Thus we can conclude, by using Theorem 5, that

$$\omega(x) = \psi^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

satisfies properties (i), (ii) and (iii).

Notice that we have proved that the likelihood can be expressed as the sum of two functions which are of the following form:

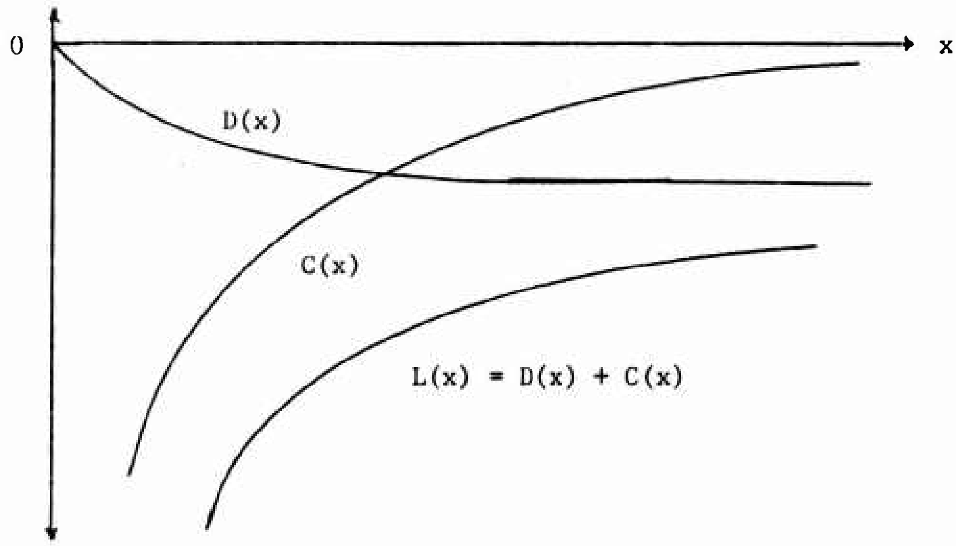

Figure 3. Likelihood function L resolved into summands C and D.

Can we conclude that $L = C + D$ has at most one maximum? We cannot, because we must assert that $L' = C' + D'$ has at most one zero using only the fact that $C' > 0$ and $D' < 0$. This conclusion, of course, is impossible based only upon these assumptions (counter examples can be easily given).

What the picture suggests is that C and D must be concave and convex respectively. If this were true it would be more likely that at most one maximum could occur. An effort to find conditions which are sufficient to insure those properties for C and D is made in the next section.

SECTION VIII

FURTHER EFFORTS TO IDENTIFY THOSE DISTRIBUTIONS FOR WHICH THE MAXIMUM LIKELIHOOD ESTIMATORS DO NOT EXIST

Before we proceed we collect some results which will be used subsequently.

Lemma 1: If ϕ is convex increasing and H is convex then $\phi(H)$ is convex and $\phi(H)$ increases or decreases accordingly as H increases or decreases.

Proof: We examine the first two derivatives of the composite function, namely

$$\phi'(H)H' \quad , \quad H''\phi'(H) + (H')^2\phi''(H) \tag{34}$$

and the result follows since $\phi' \geq 0$, $\phi'' > 0$ and $H'' \geq 0$. $||$

We have the immediate

Corollary 1: If ϕ is concave increasing and H concave then $\phi(H)$ is concave and decreases or increases accordingly as H decreases or increases.

Proof: by assumption $\phi' > 0$, $\phi'' \leq 0$, $H' \leq 0$ and the results follow. $||$

Lemma 2: If ϕ is convex decreasing and H concave increasing then $\phi(H)$ is convex decreasing.

Proof: Since $\phi' < 0$, $\phi'' \geq 0$, $H' > 0$, $H'' \leq 0$, we merely examine the expressions in (34). $||$

Lemma 3: If ϕ is any concave decreasing function on $(0,\infty)$, i.e. $\phi'' \le 0$, $\phi' \le 0$, then $\phi(|y|/\gamma)$ is concave increasing for $\gamma > 0$.

Proof: Again it is sufficient to examine the first and second derivatives:

$$\phi'(|y|/\gamma)\left(\frac{-|y|}{\gamma^2}\right), \quad \phi'(|y|/\gamma)\frac{2|y|}{\gamma^3} + \left(\frac{-|y|}{\gamma^2}\right)^2 \phi''(|y|/\gamma)$$

from which the contention is clear. $||$

Corollary 2: If $\phi(x) = \frac{\omega(x)}{x}$ is decreasing and ω concave for $x > 0$ then $h(\gamma) = |y|\ \phi(|y|/\gamma)$ for $\gamma > 0$ is concave increasing.

Proof: Clearly $h(\gamma)$ is increasing since $\phi' < 0$ and

$$h'(\gamma) = |y|\phi'(|y|/\gamma)\left(\frac{-|y|}{\gamma^2}\right) > 0 .$$

Now since

$$h''(\gamma) = |y|\left[\phi'(|y|/\gamma)\left(\frac{2|y|}{\gamma^3}\right) + \left(\frac{-|y|}{\gamma^2}\right)^2 \phi''(|y|/\gamma)\right]$$

we see $h'' < 0$ iff

$$2x\phi'(x) + x^2\phi''(x) < 0 \qquad \text{for all } x > 0 .$$

Since

$$\phi' = \frac{\omega'}{x} - \frac{\omega}{x^2} \qquad \text{and} \qquad \phi'' = \frac{\omega''}{x} - \frac{2\omega'}{x^2} + \frac{2\omega}{x^3}$$

one checks that $h'' < 0$ iff $x\,\omega'' < 0$. $||$

We now examine several different sets of assumptions concerning the behavior of ω which show that the likelihood function when μ is fixed may not have a maximum as a function of $\gamma > 0$. That is to say the maximum likelihood estimates may not exist for all three parameters jointly.

Let us consider the two assumptions:

(iv) $\omega'(x)$ is concave for $x > 0$

(v) $\omega'(1/x)$ is log-concave for $x > 0$.

One checks that neither of these assumptions implies the other.

From assumption (i) we know that $\phi(x) = \frac{\omega(x)}{x}$ is decreasing on $(0,\infty)$ since $x\phi' = \omega - \frac{\omega}{x} < 0$. By assumption (ii) we know ω is concave on $(0,\infty)$, and thus by Corollary 2 it follows that for each $i = 1,\ldots n$ the function

$$h_i(\gamma) = |y_i| \quad \phi(|y_i|/\gamma) \qquad \text{is concave increasing.}$$

Moreover since $h_i > 0$ we know h_i^2 is also concave increasing. Thus

$$\frac{1}{n}\sum_1^n h_i(\gamma) \qquad \text{is concave increasing.}$$

Now since $-\frac{1}{2}\ln x$ is convex decreasing we conclude by Lemma 2 that

$$A(\gamma) = -\frac{1}{2}\ln\left[\frac{1}{n}\sum_1^n h_i^2(\gamma)\right] \qquad \text{is convex decreasing.} \tag{35}$$

Moreover we see it has limits of

$$A(\infty) = -\frac{1}{2}\ln\left[\frac{1}{n}\sum_1^n |y|^2\right] , \qquad A(0+) = A(\infty) - \ln\omega'(\infty) .$$

By assumption (iv) we also know that $\omega'(x)$ is concave decreasing. Thus by Lemma 3, $H(\gamma) = \omega'(|y_i|/\gamma)$ is concave increasing. But $\ln x$ is concave increasing, hence by Corollary 1 it follows that

$$B(\gamma) = \frac{1}{n}\sum_1^n \ln[\omega'(|y_i|/\gamma)] \tag{36}$$

is concave and increases to zero.

Now we state

Theorem 11: Under assumptions (i), (ii), (iii) and (iv) the likelihood function defined in (2), with fixed, is of the form

$$L(\gamma) = A(\gamma) + B(\gamma) \quad \text{for } \gamma > 0$$

where A, as defined in (35), is convex decreasing to a finite negative limit and B, as defined in (36), is concave increasing to zero and moreover L has no maximum.

We defer the proof until later and state

Theorem 12: If assumption (iv) is replaced by (v) in Theorem 11 the conclusion remains the same.

Proof: We have only to check that (v) implies that the functions defined for $\gamma > 0$, for $i = 1,\ldots,n$ by

$$\ln \omega'(|y_i|/\gamma)$$

are concave increasing to zero and hence $B(\gamma)$ is also concave increasing to zero. $||$

We now give the additional details for the proof of Theorem 11. Since $L = A + B$ with A as defined in (35) and B as in (36) we find

$$L' = \frac{\sum h_i h_i'}{\sum h_i^2} - \frac{1}{n} \sum \frac{\omega''(|y_i|/\gamma)(|y_i|/\gamma^2)}{\omega'(|y_i|/\gamma)} .$$

Now $L' > 0$ iff

$$\frac{1}{n} \sum_1^n h_i h_i' > \frac{1}{n} \sum_{j=1}^n h_j^2 \; \frac{1}{n} \sum_1^n \frac{\omega''(|y_i|/\gamma)(|y_i|/\gamma^2)}{\omega'(|y_i|/\gamma)} . \tag{37}$$

Multiplying through by γ and letting

$$a_i = h_i^2 , \quad b_i = \frac{h_i'}{h_i} , \quad c_i = \frac{\omega''(|y_i|/\gamma)}{\omega'(|y_i|/\gamma)} \frac{|y_i|}{\gamma} ,$$

we can write (37) as

$$\frac{1}{n} \sum a_i b_i > \frac{1}{n} \sum a_i \cdot \frac{1}{n} \sum c_i . \tag{38}$$

But by the proof of Theorem 7 we know that equation (31) follows from assumptions (i), (ii) and (iii), namely

$$\frac{1}{n} \sum a_i b_i > \frac{1}{n} \sum a_i \frac{1}{n} \sum b_i .$$

We can establish (38) by showing that $b_i > c_i$ for $i = 1,\ldots n$ where by definition

$$1 - \zeta(|y_i|/\gamma) > \frac{\omega''(|y_i|/\gamma)}{\omega'(y_i/\gamma)} \cdot \frac{|y_i|}{\gamma} .$$

This last inequality is true if

$$1 - \zeta(x) > \frac{\omega''(x) \cdot x}{\omega'(x)} \qquad \text{for all } x > 0 .$$

but since by (iii), $1 - \zeta(x) > 0$ while also by (ii) $\omega''(x) < 0$ for $x > 0$, we have the last inequality established. ||

Let us consider again the case

$$\omega(x) = \ln(x + \sqrt{x^2+1}) . \qquad -\infty < x < \infty .$$

Now

$$\eta_\omega(y) = \lim_{x \to \infty} \frac{\omega(yx)}{\omega(x)} = y \lim_{\to \infty} \frac{\omega'(yx)}{\omega'(x)}$$

$$= y \lim_{x \to \infty} \sqrt{\frac{1 + x^2}{1 + (xy)^2}} = 1 .$$

Thus we see the function D is negative decreasing with

$$D(0+) = 0 , \qquad D(\infty) = \frac{1}{n}\sum_{1}^{n} \ln|y_i| - \frac{1}{2}\ln \frac{1}{n}\sum_{1}^{n} |y_i|^2 .$$

On the other hand

$$\delta = \lim_{x \to \infty} \zeta_\omega(x) = \lim_{x \to \infty} \frac{x\, \omega'(x)}{\omega(x)} = \lim_{x \to \infty} \frac{x/\sqrt{1+x^2}}{\omega(x)} = 0 .$$

As a consequence the negative and increasing function C has the range of values

$$C(0+) = -\infty , \qquad C(\infty) = 0 .$$

Thus $L = C + D$ and C, D appear schematically as drawn in Figure 3 .

We check that condition (iv) is not satisfied for all $x > 0$ since $\omega'''(x) < 0$ only if $0 < x < 1/\sqrt{2}$. To see this examine equation (33) . Now we check to see if condition (v) is satisfied. We must show that

$$\ln \omega'(1/x) \quad \text{is concave.}$$

Taking the second derivative and substituting from (33) we find that it is negative iff

$$\frac{-1}{x^2} \le \frac{1-x^2}{(1+x^2)^2} \quad \text{iff} \quad -1 \le 3x^2 \quad \text{for } x > 0 .$$

SECTION IX

CONCLUSION

In this report we have examined the fatigue process and broken it down into three distinct intervals each with separate behavioral properties. This model with its resulting complexity has been adopted as an alternative to the usual stochastic models with two-dimensional parametric distributions or the deterministic models involving crack growth as a solution of a differential equation which has been formulated from the assumptions of fracture mechanics.

A three-dimensional parametric model was synthesized which incorporated, as far as possible, the main features of all distinct phases. The third parameter, called the flexure parameter, is an alternative to introducing a location parameter in a two-dimensional model, such as the Wiebull, before which the probability of failure is zero.

The main objection to the introduction of a location parameter as "safe life" is that if such a time does not exist, in fact, then serious errors can be made in the determination of the initial inspection periods.

The three-dimensional model which was adopted here includes, as special cases, many of the simpler two-dimensional models previously studied. Naturally, such a general class must be carefully substantiated and the estimation techniques, in order to be useful in the specification of reliability performance, must be derived for the types of fatigue data which are available.

One of the surprises of this investigation is that there is a large class of such distributions which conform with the extant data for which all three parameters cannot be simultaneously estimated by the statistical method of maximum likelihood.

The study of this last section was completed to discover those conditions under which the derivative of the likelihood function for the unknown flexure parameter γ, when the other two parameters are unknown, has no zero. As an important practical consequence we are faced with the situation where a complete determination of the parameters determining the fatigue process cannot be accomplished by maximum likelihood estimation techniques. Thus certain of the parameters must be calculated by methods which rely upon knowledge from other disciplines. Hence those disposable constants which appear in the model must be related by theory to constants determined from the material or the geometry of the specimens.

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LIST OF SYMBOLS

β	the parameter called characteristic fatigue life
K	the regression equation for characteristic life as a function of maximum stress: the Wöhler equation
N	the random number of cycles until fatigue failure
S_n	the total random crack length at the nth cycle
X_i	the random incremental crack growth for ith cycle
μ, σ^2	the mean and variance of X_i
Φ	the standard normal distribution
F,G	(with or without affixes) generic symbols for distributions
f,g	density corresponding to upper case distribution
α	shape parameter
ω	(with or without affixes) the critical number of cycles until failure
ξ	the disposable function expressing the reciprocal relationship
T	(with or without affixes) the random time for a phase of crack growth
S(t)	the stochastic crack length at time t
U	the ξ-normal variate expressing the rate of crack growth

$p(t : a,b)$	the propagation factor determined as the solution of a differential equation from principles of fracture mechanics
A	the ξ-normal variate controlling the random time in the third phase of crack growth
Z	the standard normal variate
$\psi = \xi^{-1}$	
μ,γ,α	the location, scale and flexure parameter for the log-life model
$\omega(x) = \xi(e^x)$	the disposable function for the log-life variates
L	the likelihood function
P,R	product and ratio functions of certain derivatives of ω
C,D and A,B	decompositions of the likelihood function
ζ_ω	product of certain derivation of ω indicative of the degree of s-shapedness
h	a function defined using a reciprocal of the argument in $\omega(x)/x$

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